

# Grassmannian and Elliptic Operators

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**Abstract.** *A multi-dimensional analogue of the Krichever construction is discussed.*

## 1. Introduction

Infinite-dimensional Grassmannian (the Sato Grassmannian) plays an important role in mathematical physics, as well as in other branches of mathematics. It was conjectured, in particular, that it constitutes a natural framework for non-perturbative formulation of the string theory [1–5]. This conjecture is based first of all on the remark that moduli spaces of algebraic curves of all genera are embedded in the Grassmannian by means of so called Krichever construction. New evidence in favor of this conjecture was presented recently in [6].

The development of the string theory led to a conclusion that strings should not be considered as fundamental objects. They should appear in broader theory on equal footing with their multidimensional analogues – membranes. If we believe that this broader theory can be formulated in terms of infinite-dimensional Grassmannians, we can conjecture that membranes are related to the Grassmannian by means of multidimensional generalization of the Krichever construction. The main goal of this paper is to describe such a generalization.

Using the well-known relation between the  $\bar{\partial}$ -operator and the Dirac operator we can formulate the Krichever construction in terms of the Dirac operator; its multidimensional generalization also can be formulated in this way.

Namely, we shall consider a Riemannian manifold  $M$  with boundary  $\Gamma$ , a Hermitian vector bundle  $E$  over  $M$ , and a connection in this bundle. These data permit us to construct corresponding Dirac operator  $A$  (if some topological conditions are satisfied). In physicist's terms, we consider the Dirac operator in an external gauge field. This operator acts in the space of smooth sections of the vector bundle  $S \otimes E$  where  $S$  stands for the spinor bundle over  $M$ . One can introduce a natural Hermitian inner product in this space of sections, and one can construct a Hilbert space as the completion of the space of smooth sections with respect to this inner product. The Dirac operator becomes a self-adjoint operator in this Hilbert space. One can also introduce the boundary Hilbert space  $\mathcal{H}_\Gamma$ , the completion of the space of smooth sections of  $S \otimes E$  over  $\Gamma$ . We define the subspace  $H_+^A$  of  $\mathcal{H}_\Gamma$  as the

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\* Partially supported by NSF

closure of the space of all sections of  $S \otimes E$  over  $\Gamma$  that can be extended to smooth solutions of the Dirac equation  $Af = 0$  over  $M$ . Using the decomposition of  $\mathcal{H}_\Gamma$  in the direct sum of  $H_+^A$  and its orthogonal complement,  $H_-^A$ , we construct the Segal–Wilson version of the Sato Grassmannian  $Gr$  in the standard way: a linear subspace  $V \subset \mathcal{H}_\Gamma$  belongs to  $Gr$  if the projection  $\pi_+ : V \rightarrow H_+^A$  is a Fredholm operator and the projection  $\pi_- : V \rightarrow H_-^A$  is a compact operator. Here  $\pi_\pm$  is the orthogonal projection onto  $H_\pm^A$ .

Let us replace now the manifold  $M$  by a manifold  $\tilde{M}$  that has the same boundary  $\Gamma$ , and, instead of the Hermitian vector bundle  $p_E : E \rightarrow M$ , let us take a Hermitian vector bundle  $p_{\tilde{E}} : \tilde{E} \rightarrow \tilde{M}$ . The new objects are taken in such a way that they coincide with old ones on  $\Gamma$ . More accurately,

- (i) Riemannian metrics on  $\Gamma$  that are induced by Riemannian metrics on  $M$  and  $\tilde{M}$  coincide;
- (ii) there exist a neighborhood  $U$  of  $\Gamma$  in  $M$ , a neighborhood  $\tilde{U}$  of  $\Gamma$  in  $\tilde{M}$ , and an isomorphism  $\Psi : \tilde{E}_{\tilde{U}} \rightarrow E_U$  (here  $E_U$  is the restriction of  $E$  to  $U$ ,...) such that
  - (a)  $p_{\tilde{E}} = p_E \circ \Psi$  over  $\Gamma$ ;
  - (b)  $\Psi$  is an isometry over  $\Gamma$  (when restricted to  $\tilde{E}_\Gamma$ );
  - (c)  $\Psi^{-1}\nabla\Psi - \tilde{\nabla}$  is a differential operator of order 0 on  $\Gamma$ . Here  $\nabla$  and  $\tilde{\nabla}$  are connections on  $E$  and  $\tilde{E}$ , respectively.

One can define new Dirac operator  $\tilde{A}$ , and this operator gives rise to the subspace  $H_+^{\tilde{A}}$  of  $\mathcal{H}_\Gamma$ . We will prove that  $H_+^{\tilde{A}}$  belongs to the Grassmannian. More precisely,  $\pi_+ : H_+^{\tilde{A}} \rightarrow H_+^A$  is a Fredholm operator of index 0, and the operator  $\pi_- : H_+^{\tilde{A}} \rightarrow H_-^A$  belongs to the Schatten ideal  $\Sigma_p$  for  $p > \dim M - 1$ .

The spinor bundle  $S$  over  $M$  can be decomposed into the direct sum of the bundle of left spinors,  ${}^L S$ , and the bundle of right spinors,  ${}^R S$ . The Dirac operator takes a section of  ${}^L S \otimes E$  to a section of  ${}^R S \otimes E$  and vice versa. We denote by  ${}^{L(R)}\mathcal{H}_\Gamma$  the space of smooth sections of  ${}^{L(R)}S \otimes E$ , completed with respect to the inner product that was discussed earlier. This decomposition gives us decompositions of the subspaces  $H_\pm^A$  into direct sums  $H_\pm^A = {}^L H_\pm^A \oplus {}^R H_\pm^A$ . Clearly,  ${}^L \mathcal{H}_\Gamma = {}^L H_+^A \oplus {}^L H_-^A$ . In the same way as we defined the Grassmannian of subspaces in  $\mathcal{H}_\Gamma$ , one defines the Grassmannian in  ${}^L \mathcal{H}_\Gamma$ , and  ${}^L H_+^{\tilde{A}}$  belongs to this Grassmannian. However, the index of the Fredholm operator  $\pi_+ : {}^L H_+^{\tilde{A}} \rightarrow {}^L H_+^A$  is not necessarily zero.

In the case when  $\dim M = 2$ , one can identify the spaces  ${}^L H_+^{\tilde{A}}$  with the points in  $Gr$  that can be obtained from the standard Krichever construction. Notice that in this case the operators  $\pi_- : {}^L H_+^{\tilde{A}} \rightarrow {}^L H_-^A$  belong, in particular, to the Schatten ideal  $\Sigma_2$ , i.e. they are Hilbert–Schmidt operators. The corresponding points in the Grassmannian have an interpretation in terms of the fermionic Fock space. As we mentioned earlier, in the case when the dimension of  $M$  is arbitrary, these operators belong to the Schatten ideal  $\Sigma_p$ , with  $p > \dim M - 1$ . The fermionic interpretation of the corresponding points in the Grassmannian was analyzed in [7].

The points in the Grassmannian that can be obtained by means of the Krichever construction have “large stabilizers” in appropriate groups acting on the Grassmannian, and they can be characterized by this property. This fact plays an important role in [6]. It would be interesting to obtain similar results for the multidimensional analogue of the

Krichever construction. We have only tentative results in this direction.

In the above statements, one can replace the multi-dimensional Dirac operator by an arbitrary elliptic differential operator. We will prove our main result in this generality. The proof uses standard technique of the theory of pseudodifferential operators. First, the proof will be carried out under an additional assumption that the Agmon–Seeley condition is satisfied. Then we will show that this assumption can be removed. In the case of the Dirac operator, one can use the considerations of [8] for deriving our results. Our general proof is based on the technique of [9], and it follows [9] rather closely.

The authors wish to thank G.Henkin, M.Kontsevich, and M.Shubin for discussions.

It is a privilege for us to publish our paper in this volume. J.Stasheff made outstanding contribution to mathematics. The algebraic structures that he introduced and studied play an important role in modern mathematical physics. We dedicate our paper to J.Stasheff.

## 2. A manifold without boundary

Let  $E \rightarrow M$  be a vector bundle of rank  $r$  over a compact, orientable, closed manifold  $M$  of dimension  $n$ , and let  $\Gamma$  be a hypersurface in  $M$  that divides  $M$  into two connected components,  $M_+$  and  $M_-$ . We denote by  $E_\Gamma$  the restriction of  $E$  to  $\Gamma$ , and  $E_\pm$  are restrictions of  $E$  to  $M_\pm$ . One can find a function,  $x_n$ , defined in a neighborhood of  $\Gamma$  in  $M$  such that  $x_n = 0$  on  $\Gamma$ ,  $\pm x_n > 0$  in  $M_\pm$ , and  $dx_n \neq 0$ . When local coordinates in a neighborhood of a point from  $\Gamma$  are used, they will be always assumed to be of the form  $(x', x_n)$  where  $x' = (x_1, \dots, x_{n-1})$  are local coordinates on  $\Gamma$ . We introduce a connection  $\nabla$  on the restriction of  $E$  to a neighborhood of  $\Gamma$ , and, with a slight abuse of notations, we will denote by  $\partial_n$ , or by  $\partial/\partial x_n$ , the covariant derivative with respect to  $x_n$ .

Let  $A$  be an elliptic pseudo-differential operator of order  $k > 0$  that acts on sections of  $E$ . Throughout this section, we will assume that

(i) the operator  $A$  satisfies the Agmon–Seeley condition: there exists an angle  $\{z : |\arg z - \theta| < \epsilon\}$  in the complex plane that is free from eigenvalues of the principal symbol of  $A$ ;

(ii) in a neighborhood of  $\Gamma$ , the operator  $A$  is differential;

(iii) the restriction of  $Au$  to  $M_\pm$  depends on the restriction of  $u$  to  $M_\pm$  only.

For technical reasons, we also assume that 0 is a regular point of  $A$ . This assumption is not essential. We will indicate, what changes should be made in the case when the operator  $A$  is not invertible. A neighborhood of  $\Gamma$  in  $M$  is diffeomorphic to  $\Gamma \times (-1, 1)$ . We fix this diffeomorphism once and forever;  $x_n$  is the coordinate along the interval  $(-1, 1)$ . Assume that in this neighborhood our operator  $A$  is differential, and it can be written as

$$A = A_k \partial_n^k + \cdots + A_1 \partial_n + A_0$$

where  $A_q$  is a differential operator of order  $k - q$  that contains the tangential derivatives only. In particular  $A_k$  is a differential operator of order 0, that is a smooth family of endomorphisms of fibers of  $E$ . Ellipticity of the operator  $A$  implies that the endomorphisms  $A_k$  are non-degenerate. In fact, if one denotes by  $\xi'$  the set of dual variables to  $x'$ , and by  $\xi_n$  the variable dual to  $x_n$  then the principal symbol of  $A$ , when evaluated at a point  $(x; \xi' = 0, \xi_n)$ , equals  $A_k(x)(i\xi_n)^k$ . It should be non-degenerate when  $\xi_n \neq 0$ .

We introduce two subspaces,  $\mathcal{L}_\pm$ , of the space

$$\mathcal{L} = \underbrace{C^\infty(\Gamma, E_\Gamma) \oplus \cdots \oplus C^\infty(\Gamma, E_\Gamma)}_{k \text{ times}}$$

of sections of the vector bundle

$$\mathcal{E} = \underbrace{E_\Gamma \oplus \cdots \oplus E_\Gamma}_{k \text{ times}}$$

in the following way

$$\mathcal{L}_\pm = \{(\phi_0, \dots, \phi_{k-1}) : \phi_j = \partial_n^j u \quad \text{where} \quad Au = 0 \quad \text{in} \quad M_\pm\}.$$

**Proposition 1.** *Under all assumptions made above,*

$$\mathcal{L}_+ \cap \mathcal{L}_- = \{0\}; \quad \mathcal{L}_+ + \mathcal{L}_- = \mathcal{L}.$$

**Proof.** The first statement is almost obvious. Let  $\phi = (\phi_0, \dots, \phi_{k-1}) \in \mathcal{L}_+ \cap \mathcal{L}_-$ . Then there exist sections  $u_\pm$  of  $E_\pm$  such that  $Au_\pm = 0$  and  $\partial_n^j u_\pm = \phi_j$ ,  $j = 0, \dots, k-1$ , on  $\Gamma$ . Then, from the equation  $Au_\pm = 0$ , it follows that all partial derivatives of  $u_+$  and  $u_-$  agree on  $\Gamma$  (we have used ellipticity of the operator  $A$ ). Therefore the section  $u$  of  $E$  defined as  $u_+$  over  $M_+$  and as  $u_-$  over  $M_-$  is smooth; so it satisfies  $Au = 0$ . Because 0 does not belong to the spectrum of  $A$ , we conclude that  $u = 0$ , and  $\phi = 0$ .

We proceed now to proving the second statement of the Proposition. Let  $u_\pm$  be a section of  $E_\pm \rightarrow M_\pm$ , smooth up to  $\Gamma$ . By  $u_\pm^0$  we denote the section of  $E \rightarrow M$  that equals  $u$  over  $M_\pm$ , and that equals 0 over  $M_\mp$ . Then, for every positive integer number  $q$ ,

$$\partial_n^q (u_\pm^0) = (\partial_n^q u_\pm)^0 \pm \sum_{p=0}^{q-1} \phi_{q-p-1}^\pm \delta^{(p)}(x_n)$$

where  $\phi_j^\pm$  is the restriction of  $\partial_n^j u_\pm$  to  $\Gamma$ . We will denote by  $\mathcal{A}_q$  restrictions of operators  $A_q$  to  $\Gamma$ . They are differential operators acting on sections of the vector bundle  $E_\Gamma \rightarrow \Gamma$ . One has

$$\begin{aligned} A(u_\pm^0) &= (Au_\pm)^0 \pm \sum_{q=1}^k \mathcal{A}_q \sum_{p=0}^{q-1} \phi_{q-p-1}^\pm \delta^{(p)}(x_n) \\ &= (Au_\pm)^0 \pm \sum_{p=0}^{k-1} \left( \sum_{q=p+1}^k \mathcal{A}_q \phi_{q-p-1}^\pm \right) \delta^{(p)}(x_n) \\ &= (Au_\pm)^0 \pm \sum_{p=0}^{k-1} \psi_p^\pm \delta^{(p)}(x_n). \end{aligned} \tag{1}$$

The section  $\psi_\pm = (\psi_\pm^0, \dots, \psi_\pm^{k-1})$  of  $\mathcal{L}$  is related to the section  $\phi_\pm$  by

$$\psi_\pm = \mathcal{A}\phi_\pm$$

where  $\mathcal{A}$  is a differential operator acting on sections of  $\mathcal{L}$ , and it has block representation

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 & \dots & \mathcal{A}_{k-1} & \mathcal{A}_k \\ \mathcal{A}_2 & \mathcal{A}_3 & \dots & \mathcal{A}_k & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{A}_k & 0 & \dots & 0 & 0 \end{pmatrix}. \quad (2)$$

Clearly, the operator  $\mathcal{A}$  is invertible; its inverse is of the form

$$\mathcal{A}^{-1} = \begin{pmatrix} 0 & \dots & 0 & \mathcal{A}_k^{-1} \\ 0 & \dots & \mathcal{A}_k^{-1} & -\mathcal{A}_k^{-1}\mathcal{A}_{k-1}\mathcal{A}_k^{-1} \\ \vdots & \ddots & \vdots & \vdots \end{pmatrix}.$$

We recall that  $\mathcal{A}_k$  is a smooth family of automorphisms.

Now we are ready to prove  $\mathcal{L}_+ + \mathcal{L}_- = \mathcal{L}$ . Take a section  $\phi = (\phi_0, \dots, \phi_{k-1})$  from  $\mathcal{L}$ . Let  $\psi = (\psi_0, \dots, \psi_{k-1}) = \mathcal{A}\phi$ , and let  $\Psi$  be a distribution

$$\Psi = \sum_{p=0}^{k-1} \psi_p \delta^{(p)}(x_n). \quad (3)$$

Set  $u = A^{-1}\Psi$ , and let  $u_\pm$  be the restriction of  $u$  to  $M_\pm$ . It follows from the elliptic regularity theory that  $u_\pm$  are smooth sections of  $E_\pm$ , up to  $\Gamma$ . Clearly,  $Au_\pm = 0$  in  $M_\pm$ . Denote by  $\phi_j^\pm$  the restriction of  $\partial_n^j u_\pm$  to  $\Gamma$ . Then sections  $\phi^\pm = (\phi_1^\pm, \dots, \phi_{k-1}^\pm)$  belong to  $\mathcal{L}_\pm$ , and (1) implies that

$$\psi = \mathcal{A}\phi^+ - \mathcal{A}\phi^-.$$

Because of invertibility of  $\mathcal{A}$ , we conclude that  $\phi = \phi^+ - \phi^-$ .

○

Instead of the space  $\mathcal{L}$  of smooth sections of  $\mathcal{E}$ , one can take a Hilbert space

$$\mathcal{H} = H^{k-1+\alpha}(\Gamma, E_\Gamma) \oplus \dots \oplus H^{1+\alpha}(\Gamma, E_\Gamma) \oplus H^\alpha(\Gamma, E_\Gamma)$$

of sections. Here  $\alpha$  is a sufficiently large positive number. Let  $\mathcal{H}_\pm$  be the closure of  $\mathcal{L}_\pm$  in  $\mathcal{E}$ . It follows immediately from Proposition 1 and from the standard elliptic estimates that

$$\mathcal{H}_+ \cap \mathcal{H}_- = \{0\} \quad \text{and} \quad \mathcal{H}_+ + \mathcal{H}_- = \mathcal{H}.$$

To define Sobolev spaces of sections of the vector bundle  $E_\Gamma$ , one needs some additional structures: a Riemannian metric on  $\Gamma$ , a Hermitian structure on  $E_\Gamma$ , and a connection  $\nabla^\Gamma$  on  $E_\Gamma$ . A Riemannian metric and a Hermitian structure give rise to the  $L^2$ -scalar product on both  $C^\infty(\Gamma, E_\Gamma)$  and  $\Lambda^1(\Gamma, E_\Gamma)$ , the space of sections of  $E_\Gamma$  and the space of one-forms with values in  $E_\Gamma$ . In the usual way, a connection induces the operator

$$d^\nabla : C^\infty(\Gamma, E_\Gamma) \rightarrow \Lambda^1(\Gamma, E_\Gamma)$$

by the formula

$$d^\nabla = \sum \nabla_{x_i}^\Gamma dx_i$$

in local coordinates. The Laplacian on the space of sections of  $E_\Gamma$  can be defined as  $\Delta = (d^\nabla)^* d^\nabla$ . Then, the  $H^s$ -scalar product is defined as

$$(\phi, \psi)_s = ((\Delta + 1)^s u, v)_{L^2}.$$

Of course, the actual formula for the scalar product depends on all choices made but it is a well-known fact that spaces themselves are independent of these choices.

**Proposition 2.** *Let  $R_\pm$  be the projection onto  $\mathcal{H}_\pm$  parallel to  $\mathcal{H}_\mp$ . The operators  $R_\pm$  are pseudodifferential operators, and their complete symbols depend only on coefficients of  $A$  and on their derivatives on the hypersurface  $\Gamma$ .*

**Proof.** We will treat the projector  $R_+$ ; clearly, the case of  $R_-$  is similar. Let us define a new operator  $\tilde{R}$  acting on sections of  $\mathcal{E}$ . We describe now how to construct  $\zeta = (\zeta_0, \dots, \zeta_{k-1}) = \tilde{R}\phi$  where  $\phi = (\phi_0, \dots, \phi_{k-1})$ . Firstly, we define  $\psi = \mathcal{A}\phi$  (see (2) for  $\mathcal{A}$ ), then the section  $\psi$  is used to produce the distribution  $\Psi$  (see (3)), then the section  $u$  of  $E$  is defined by  $u = A^{-1}\Psi$ , and, finally,  $\zeta_j$  is the restriction to  $\Gamma$  of  $\partial_n^j u_+$  where  $u_\pm$  are the restrictions of  $u$  to  $M_\pm$ . We will show that, in fact,  $\tilde{R} = R_+$ , and then we will see that  $\tilde{R}$  is a pseudodifferential operator, and we will discuss how to compute its symbol.

To show that  $\tilde{R} = R_+$  one has to verify that  $\tilde{R}\phi = \phi$  when  $\phi \in \mathcal{H}_+$  and  $\tilde{R}\phi = 0$  when  $\phi \in \mathcal{H}_-$ . Let  $\phi \in \mathcal{H}_+$ . Then there exists a solution  $v$  of the equation  $Av = 0$  in  $M_+$  such that  $\phi_j$  is the restriction to  $\Gamma$  of  $\partial_n^j v$ . Let  $v^0$  be the section of  $E$  that equals  $v$  over  $M_+$ , and that equals 0 over  $M_-$ . Then  $Av^0 = \Psi$ . Hence,  $u = v^0$ ,  $u_+ = v$ , and  $\zeta_j = \phi_j$ .

Now, let  $\phi \in \mathcal{H}_-$ . Then there exists a solution  $w$  of the equation  $Aw = 0$  in  $M_-$  such that  $\phi_j$  equals the restriction of  $-\partial_n^j w$  to  $\Gamma$ . Let  $w^0$  be the section of  $E$  that coincides with  $w$  over  $M_-$ , and that vanishes over  $M_+$ . Then  $Aw^0 = \Psi$ , so  $w = u$ , and  $u_+ = 0$ . We conclude that  $\zeta = \tilde{R}\phi = 0$ .

Denote by  $r^+$  the operator of restricting a section from  $M_+$  to  $\Gamma$ . Let  $B_{qp}$  be the operator acting on sections of  $E_\Gamma$  according to the formula

$$B_{qp}\psi_p = r^+ \partial_n^q A^{-1}(\psi_p \delta^{(p)}(x_n)).$$

Let  $\mathcal{B}$  be the operator acting on sections of  $\mathcal{E}$ , the block-matrix of which is  $(B_{qp})$ . Clearly,

$$R_+ = \tilde{R} = \mathcal{B}\mathcal{A}. \quad (4)$$

Denote by  $S(x', x_n; \xi', \xi_n)$  the symbol of  $A^{-1}$  in certain local coordinate system, with respect to a local frame. Up to a smoothing operator applied to  $\psi_p$ ,  $r + \partial_n^q A^{-1}(\psi_p(x') \delta^{(p)}(x_n))$  equals

$$\lim_{x_n \rightarrow 0^+} (2\pi)^{-n} i^{p+q} \int \xi_n^{p+q} e^{i(x' - y')} \xi' e^{ix_n \xi_n} S(x', x_n; \xi', \xi_n) dy' d\xi' d\xi_n.$$

It is a well known fact that such an operator is a pseudodifferential operator (e.g. see [10]), and its symbol equals

$$\lim_{x_n \rightarrow 0^+} \frac{i^{p+q}}{2\pi} \int_{-\infty}^{\infty} \xi_n^{p+q} e^{ix_n \xi_n} S(x', x_n; \xi', \xi_n) d\xi_n. \quad (5)$$

The complete asymptotic symbol  $S$  is meromorphic in  $\xi_n$ , and the expression (5) can be rewritten as

$$\sigma(B_{qp}) = \frac{i^{p+q}}{2\pi} \int_{\gamma_+} S(x', 0; \xi', \xi_n) d\xi_n \quad (6)$$

where  $\sigma$  means “symbol”, and  $\gamma_+$  is a contour in the complex  $\xi_n$ -plane that goes around all poles of  $S$  in the counter-clockwise direction. The symbol of  $\mathcal{B}$  depends only on the restriction of the symbol of  $A^{-1}$  to  $\Gamma$ , and this restriction depends only on coefficients of  $A$  and on their derivatives on  $\Gamma$ .

○

Matrix elements,  $(R_+)_q{}_j$ , of the operator  $R_+$  are given by

$$(R_+)_q{}_j = \sum_{p=0}^{k-1-j} B_{qp} \mathcal{A}_{p+j+1} \quad (7)$$

(cf. (2) and (4)). It can be easily seen from (6) or (5) that the order of  $B_{qp}$  equals  $p + q - k + 1$ . The order of the operator  $\mathcal{A}_{p+j+1}$  equals  $k - p - j - 1$ , so each term on the right in (7) is a pseudodifferential operator of order  $q - j$ . We conclude that

$$\text{ord}(R_+)_q{}_j = q - j.$$

It is clear from (2) and (6) that the principal symbol of  $(R_+)_q{}_j$  depends only on the value of the principal symbol of  $A$  on  $\Gamma$ .

If a pseudodifferential operator has a block-matrix form, and the order of its  $(qj)$ -entry is  $q - j$  then by a principal symbol of such an operator we mean a matrix, the  $(qj)$ th entry of which is the  $(q - j)$ -principal symbol of the corresponding operator (this is the principal symbol in the sense of Agmon–Douglis–Nirenberg). Such an operator is similar to an operator of order 0.

Now, let  $P_\pm$  be the orthogonal projector onto the space  $\mathcal{H}_\pm$ .

**Theorem 1.**  $P_\pm$  is a pseudodifferential operator; its  $(qj)$ -th entry in the block-matrix representation has order  $q - j$ . The complete symbol of  $P_\pm$  depends only on the coefficients of  $A$  and on their derivatives on  $\Gamma$ , and the principal symbol of  $P_\pm$  depends only on the value of the principal symbol of  $A$  on  $\Gamma$ .

**Proof.** Because operators  $R_\pm$  are projectors, their principal symbols are also projectors. Therefore, for any value of  $\lambda \neq 0, 1$ , the operator  $\lambda - R_\pm$  is elliptic in the sense of Agmon–Douglas–Nirenberg, and the resolvent  $(\lambda - R_\pm)^{-1}$  is a holomorphic in  $\lambda$  family of pseudodifferential operators that have the same structure. All statements of the theorem follow from the formula

$$P_\pm = \frac{1}{2\pi i} \int_{\gamma} (\lambda - R_\pm)^{-1} d\lambda$$

where  $\gamma$  is, say, a circle of radius  $1/2$  centered at the origin and taken with the counter-clockwise orientation.

○

### 3. A manifold with boundary

Now, instead of treating a closed manifold that is divided into two connected components by a hypersurface, we will consider a compact manifold  $M_0$  with smooth boundary  $\Gamma$ . Let  $E_0$  be a vector bundle over  $M_0$ , let  $A$  be an elliptic differential operator of order  $k$  acting on sections of  $E$  that satisfies the Agmon–Seeley condition. By  $\mathcal{L}_0$  we denote the subspace of  $\mathcal{L}$  that consists of sections  $(u, \partial_n u, \dots, \partial_n^{k-1} u)$  where  $Au = 0$  in  $M_0$ , and  $P_0$  is the orthogonal projection onto  $\mathcal{L}_0$ . All notations are the same as those used above, with the only difference that a neighborhood of  $\Gamma$  is diffeomorphic to  $\Gamma \times [0, 1]$  (not to  $\Gamma \times (-1, 1)$ ), and we use the subscript 0 instead of + or −. We will prove the following theorem.

**Theorem 2.**  $P_0$  is a pseudodifferential operator; its  $(qj)$ -th entry in the block-matrix representation has order  $q - j$ . The complete symbol of  $P_0$  depends only on the coefficients of  $A$  and on their derivatives on  $\Gamma$ , and the principal symbol of  $P_0$  depends only on the value of the principal symbol of  $A$  on  $\Gamma$ .

**Proof.** We will construct a closed manifold  $M \supset M_0$ , a vector bundle  $E \rightarrow M$  that extends  $E_0$ , and an extension of  $A$  to an invertible, selfadjoint, elliptic operator acting on sections of  $E$ . Then the statements of Theorem 2 will follow from Theorem 1 (just replace the subscript 0 by +).

For the manifold  $M$ , we take the double of  $M_0$ , and, for the vector bundle  $E$ , we take the double of  $E_0$ . A neighborhood of  $\Gamma$  in  $M$  is diffeomorphic  $\Gamma \times (-1, 1)$ . Clearly,  $A$  can be extended to an elliptic differential operator that satisfies the Agmon–Seeley condition, and that acts on sections of  $E$  over  $M_0 \cup \Gamma \times (-1, 0)$ . We will construct an extension of  $A$  to the whole manifold  $M$  in two steps.

**Step 1.** Let  $a(x, \xi)$  be the principal symbol of  $A$ , and let  $\gamma$  be a positively oriented contour in the complex plane that does not intersect a ray  $z = re^{i\theta}$ ,  $r \geq 0$ , and that encloses all eigenvalues of  $a(x, \xi)$  for all  $x$  and all  $\xi$ ,  $|\xi| = 1$ . Here, we have chosen an arbitrary metric on the cotangent bundle to  $M$ . The existence of such a contour is guaranteed by the Agmon–Seeley condition. Let  $\chi(\tau)$  be a smooth non-negative function of one variable that equals 0 when  $\tau < -3/4$ , and that equals 1 when  $\tau > -1/2$ . We define a symbol  $b(x, \xi)$  on the cosphere bundle  $S^*M$  (it is an endomorphism of the vector bundle  $E$ , pulled back to  $S^*M$ ) in the following way:  $b(x, \xi) = a(x, \xi)$  over  $M_0$ ,  $b(x, \xi)$  equals the identity operator over  $M \setminus (M_0 \cup \Gamma \times (-1, 0))$ , and

$$b(x, \xi) = \frac{1}{2\pi i} \int_{\gamma} z^{\chi(x_n)} (z - a(x, \xi))^{-1} dz$$

over  $\Gamma \times (-1, 0)$ . To define the powers of  $z$ , one makes the cut  $re^{i\theta} : r \geq 0$  in the complex plane. The symbol  $b(x, \xi)$  is extended to the whole cotangent bundle by  $k$ -homogeneity. Let  $B$  be a pseudo-differential operator with the principal symbol  $b(x, \xi)$ . Clearly, the operator  $B$  is elliptic, it satisfies the Agmon–Seeley condition, and its principal symbol equals  $a(x, \xi)$  over  $M_0 \cup \Gamma \times (-1/2, 0)$ .

**Step 2.** Choose smooth function  $\phi(x)$  and  $\psi(x)$  on  $M$  such that  $\phi(x) = 1$  on  $M_0 \cup \Gamma \times (-1/4, 0)$ ,  $\phi(x) = 0$  outside of  $M_0 \cup \Gamma \times (-1/2, 0)$ , and  $\phi^2(x) + \psi^2(x) = 1$ . Define the operator

$$\tilde{A} = \phi(x)A\phi(x) + \psi(x)B\psi(x).$$

The operator  $\tilde{A}$  is a pseudo-differential elliptic operator that satisfies the Agmon–Seeley condition, it is differential in  $\Gamma \times (-1/4, 1)$ , and it coincides with  $A$  over  $M_0$ . If it is invertible, then one can apply Theorem 1. If it is not invertible, then one can make it invertible by adding to it an operator of multiplying by an appropriate function supported on  $M \setminus M_0$ .

○

**Remark 1.** The Theorem 2 holds for pseudo-differential operators that are differential in a neighborhood of  $\Gamma$ . In fact, in the proof we did not use the fact that the operator  $A$  is differential outside  $\Gamma \times [0, 1)$ .

**Remark 2.** The statements of the Theorem 1 depend only on the restrictions of  $A$  to  $M_{\pm}$ . Theorem 2 shows (see the previous remark) that, for conclusions of Theorem 1, invertibility of  $A$  is not essential. If the operator  $A$  is not invertible, then, in Proposition 1,  $\mathcal{L}_+ \cap \mathcal{L}_-$  is finite-dimensional, and  $\mathcal{L}_+ + \mathcal{L}_-$  has finite codimension.

**Remark 3.** One can remove the assumption that the operator  $A$  is self-adjoint. In fact, the operator

$$A' = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$$

is a self-adjoint operator acting on sections of the vector bundle  $E \oplus E$ . The statements of the Theorem 2, when applied to the operator  $A'$ , imply the same statements for the operator  $A$ .

Now, let us go back to the situation when there are two differential operators around,  $A$  and  $\tilde{A}$ ; they act on sections of vector bundles  $p_E : E \rightarrow M$  and  $p_{\tilde{E}} \tilde{E} \rightarrow \tilde{M}$ , and  $\partial M = \partial \tilde{M} = \Gamma$ . We assume that there exists an isomorphism  $\Psi : E_U \rightarrow \tilde{E}_{\tilde{U}}$  such that  $p_E \circ \Psi = p_{\tilde{E}}$  over  $\Gamma$ . Here  $U$  is a neighborhood of  $\Gamma$  in  $M$  and  $\tilde{U}$  is a neighborhood of  $\Gamma$  in  $\tilde{M}$ . We also assume that the isomorphism  $\Psi$  maps the connection  $\tilde{\nabla}$  on  $\tilde{E}$  to the connection  $\nabla$  on  $E$ . To avoid confusion, let us say that in the general setting we use connections for completely different purposes than they were used in the context of the Dirac operator. For the Dirac operator, the connection was used to construct it. Here operators are given by their symbols, and the connection is used for the purpose of taking normal derivatives on the boundary. The isomorphism  $\Psi$  identifies  $U$  and  $\tilde{U}$ ,  $E_U$  and  $\tilde{E}_{\tilde{U}}$ , so we will think of  $E_U$  and  $\tilde{E}_{\tilde{U}}$  as being the same.

Let

$$a(x, \xi) = a_k(x, \xi) + a_{k-1}(x, \xi) + \cdots + a_0(x, \xi)$$

be the splitting of the total (complete) symbol of the operator  $A$  into homogeneous components in certain local coordinates. In this splitting, only the principal symbol  $a_k$  has an invariant meaning. We say that two operators  $A$  and  $\tilde{A}$  agree up to the order  $q$  on the boundary  $\Gamma$ , if they have the same order, and the corresponding homogeneous components,  $a_{k-j}$  and  $\tilde{a}_{k-j}$ , are equal on  $\Gamma$ , together with all their derivatives up to the order  $q - j$ , for  $j \leq \min\{k, q\}$ . Though the components  $a_{k-j}$  themselves are not defined invariantly, the property of two operators to agree up to a certain order on  $\Gamma$  does not depend on a particular choice of local coordinates. Note that Dirac operators that were discussed in the introduction agree up to the order 0 on  $\Gamma$ .

Denote by  $H_+^A$  the closure in  $\mathcal{H}$  of the space of Cauchy data for the operator  $A$  (of the space  $\mathcal{L}_0$  from the Theorem 2), and let  $H_+^{\tilde{A}}$  be the closure of the space of Cauchy data for the operator  $\tilde{A}$ . To measure, how far the space  $H_+^A$  is from the space  $H_+^{\tilde{A}}$ , we introduce orthogonal projections  $\pi_+^A$  and  $\pi_+^{\tilde{A}}$  onto these spaces. It follows from Theorem 2 that if the operators  $A$  and  $\tilde{A}$  agree up to the order  $q \geq 0$  on  $\Gamma$  then  $\pi_+^A - \pi_+^{\tilde{A}}$  is a compact operator, and it belongs to the Schatten ideal  $\Sigma_p$  for  $p > (n-1)/(q+1)$ .

From the construction of projections  $R_\pm$  and  $P_\pm$  in section 2 it follows that the first  $q+1$  terms in their complete symbols depend on the restriction of  $a_{k-j}(x, \xi)$ , and its derivatives up to the order  $q-j$ , to  $\Gamma$  where  $j \leq \min\{k, q\}$ . The same is true for the operator  $P_0$  from this section. It follows that if operators  $A$  and  $\tilde{A}$  agree up to the order  $q$  on  $\Gamma$  then  $\pi_+^A - \pi_+^{\tilde{A}}$  is a pseudo-differential operator, and the  $(ij)$ -th entry in its block matrix representation has the order  $i-j-q-1$ . It is well known that the singular numbers  $s_j$  of such an operator can be estimated

$$s_j \leq C j^{-(q+1)/(n-1)}$$

(note that the dimension of  $\Gamma$  equals  $n-1$ ), and, therefore, it belongs to  $\Sigma_p$  for  $p > (n-1)/(q+1)$ .

Let us now denote by  $P_{\tilde{A}A}$  the restriction of the projection  $\pi_+^A$  to  $H_+^{\tilde{A}}$  and by  $Q_{\tilde{A}A}$  the restriction of  $I - \pi_+^A$  to  $H_+^{\tilde{A}}$ . Then, the operator  $P_{\tilde{A}A}$  is Fredholm. In fact, its kernel coincides with  $H_+^{\tilde{A}} \cap (H_+^A)^\perp$ . The compact operator  $\pi_+^{\tilde{A}} - \pi_+^A$  is identical on this space; therefore it is finite-dimensional. The adjoint to  $P_{\tilde{A}A}$  is  $P_{AA\tilde{A}}$ , and, by the same reason, its kernel is finite dimensional. Further, if  $\pi_+^A - \pi_+^{\tilde{A}}$  belongs to the Schatten class  $\Sigma_p$  then  $Q_{\tilde{A}A} \in \Sigma_p$ . In fact,  $Q_{\tilde{A}A}$  is the restriction to  $H_+^{\tilde{A}}$  of the operator  $(I - \pi_+^A)\pi_+^{\tilde{A}} = (\pi_+^{\tilde{A}} - \pi_+^A)\pi_+^{\tilde{A}}$  which belongs to  $\Sigma_p$  because  $\Sigma_p$  is an ideal in the ring of bounded operators.

**Remark 4.** The index of the operator  $P_{\tilde{A}A}$  needs not be equal to 0. On the other hand, the index of the corresponding projections constructed for the operator  $A'$  (see Remark 3) always equals 0.

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